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On Some Harris-FKG Type Correlation Inequalities for a Non-Attractive Model

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We consider the following non-attractive one-dimensional model whose evolution is given by $\xi_{n+1}(x) = \xi_n(x+1) + \xi_n(x-1)$ (mod 2) with probability p, $= 0$ with probability $1-p$. In the present letter, we prove that some Harris-FKG type correlation inequalities hold for this model. Moreover it is shown that a correlation inequality is also correct for the more general non-attractive class.

KEY WORDS: The Harris-FKG type correlation inequality; non-attractiveness; the Domany–Kinzel model; survival probability; critical line.

In the present letter, we consider the following discrete-time process ξ_n^A at time n starting $A \subset 2\mathbf{Z}$ whose evolution satisfies:

(i) $P(x \in \xi_{n+1}^A \mid \xi_n^A) = f(|\xi_n^A \cap \{x-1, x+1\}|)$,

(ii) given ξ_n^A, the events $\{x \in \xi_{n+1}^A\}$ are independent, where

$$f(0) = 0, \qquad f(1) = p, \qquad \text{and} \qquad f(2) = 0$$

This process can be considered on a space $S = \{s = (x, n) \in \mathbf{Z} \times \mathbf{Z}_+ : x + n = \text{even}\}$, where $\mathbf{Z}_+ = \{0, 1, 2, \dots\}$. If we let $\xi_n(x) = 1$ if $x \in \xi_n$ and $= 0$ if $x \notin \xi_n$, then the above evolution can be rewritten as:

$$\xi_{n+1}^A(x) = \begin{cases} \xi_n^A(x+1) + \xi_n^A(x-1) \bmod 2 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

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When $f(2)=q$ with $0 \leqslant q \leqslant 1$, this more general class was first studied by Domany and Kinzel.[1] Concerning this class, the reader is referred to Durrett[2] in pp. 90–98, for example. In this setting, the oriented bond percolation ($q=2p-p^2$) and the oriented site percolation ($q=p$) are special cases. The mixed site-bond oriented percolation with the probability of open site α and with the probability of open bond β corresponds to the case of $p=\alpha\beta$ and $q=\alpha(2\beta-\beta^2)$.

When $0 \leqslant p \leqslant q \leqslant 1$, the process is called *attractive* and has the following nice property: if $\xi_n^A \subset \xi_n^B$, then we can guarantee that $\xi_{n+1}^A \subset \xi_{n+1}^B$ for any $n \geqslant 0$ by using an appropriate coupling. However, our model (i.e., $q=0$) is non-attractive, so it does not have the above property.

We let ξ_n^0 be our model at time n starting from the origin. Here we introduce a survival probability from the origin:

$$\theta(p) = P(\xi_n^0 \neq \emptyset \text{ for any } n \geqslant 0)$$

The sequence of events $\{\xi_n^0 \neq \emptyset\}$ is decreasing, so $\theta(p)$ is well defined. We introduce two critical probabilities as follows:

$$p_c = \sup\{p \in [0,1] : \theta(p') = 0 \text{ for any } p' \in [0,p]\}$$

$$p_c^* = \inf\{p \in [0,1] : \theta(p') > 0 \text{ for any } p' \in [p,1]\}$$

The above definitions give

$$0 \leqslant p_c \leqslant p_c^* \leqslant 1$$

since $\theta(0)=0$ and $\theta(1)=1$. Note that it is not known whether or not $\theta(p)$ is non-decreasing function in p, since the model under consideration is not attractive. However Monte Carlo simulations suggest that the above monotonicity is valid. That is, it is conjectured that $p_c = p_c^*$. The estimated value is $p_c \approx 0.82$ by Kinzel[3] using finite size scaling calculations.

Define the survival probability from finite set $A \subset 2\mathbf{Z}$ as

$$\sigma(A) = P(\xi_n^A \neq \emptyset \text{ for any } n \geqslant 0)$$

So $\theta(p) = \sigma(\{0\})$. By using the Harris lemma (see Harris[4]) Konno[5] gave the following first and second upper bounds ($h^{(1)}(A)$ and $h^{(2)}(A)$ respectively) on $\sigma(A)$ for finite A.

Theorem 1. Let $p_c^{(1)}=0.5$. For any $p \in [p_c^{(1)}, 1]$,

$$\text{(a)} \quad \sigma(A) \leqslant h^{(1)}(A) = 1 - \left(\frac{1-p}{p}\right)^{2|A|} \qquad \text{for all} \quad A \in Y$$

In particular, if we take $A = \{0\}$, then we have

$$\text{(b)} \quad \theta(p) \leqslant h^{(1)}(\{0\}) = 1 - \left(\frac{1-p}{p}\right)^2 \qquad (p \in [p_c^{(1)}, 1])$$

Moreover

$$\text{(c)} \quad p_c^{(1)} = 0.5 \leqslant p_c^*$$

Let $b(A)$ be the number of neighboring pairs in A.

Theorem 2. Let

$$p_c^{(2)} = \inf\{p \in [0, 1] : 2p^3 - 2p^2 + 2p - 1 \geqslant 0\} = 0.647799...$$

For any $p \in [p_c^{(2)}, 1]$, we have

$$\text{(a)} \quad \sigma(A) \leqslant h^{(2)}(A) = 1 - \alpha_*^{|A|} \beta_*^{b(A)} \qquad \text{for all} \quad A \in Y$$

where

$$\alpha_* = \frac{p^4 - 2p^3 + 2p^2 - 2p + 1}{p^4} \qquad \text{and} \qquad \beta_* = \left(\frac{p\alpha_* + 1 - p}{\alpha_*}\right)^2$$

In particular, if we take $A = \{0\}$, then we have

$$\text{(b)} \quad \theta(p) \leqslant h^{(2)}(\{0\}) = \frac{2p^3 - 2p^2 + 2p - 1}{p^4} \qquad (p \in [p_c^{(2)}, 1])$$

Moreover,

$$\text{(c)} \quad p_c^{(2)} = 0.647799... \leqslant p_c^*$$

On the other hand, concerning upper bound on p_c^*, Bramson and Neuhauser[6] proved that

$$p_c^* < 1$$

by using a rescaling argument. So the existence of the phase transition is established rigorously. Very recently, Konno, Sato and Sudbury[7] gave the following improved lower bounds on p_c and p_c^* by the method for finding suitable supermartingales for the model.

Theorem 3. In our model, we have $0.771 \leqslant p_c$, and $0.781 \leqslant p_c^*$.

This result will be used to prove some correlation inequalities in Theorem 4.

Konno[5] discussed about correlation inequalities which give the same upper bounds, i.e., $h^{(1)}(\{0\})$ and $h^{(2)}(\{0\})$, on survival probability $\theta(p)$ as follows (General arguments on correlation inequalities of interacting particle systems appeared in Konno,[8] for example.)

To explain this story, we introduce the following notation. For any finite $A \subset 2\mathbf{Z}$, we define extinction probability starting from A by

$$\nu(A) = 1 - \sigma(A) = P(\xi_n^A = \phi \text{ for some } n \geqslant 0)$$

Furthermore we let

$$\nu(\circ) = \nu(\{0\}), \qquad \nu(\circ\,\circ) = \nu(\{0, 2\}), \qquad \nu(\circ * \circ) = \nu(\{0, 4\}),$$
$$\nu(\circ\,\circ\,\circ) = \nu(\{0, 2, 4\}), \text{ etc.}$$

Note that $\theta(p) + \nu(\circ) = 1$. From the definition of this model, we have

$$\nu(\circ) = p^2 \nu(\circ\,\circ) + 2p(1-p)\,\nu(\circ) + (1-p)^2 \tag{1}$$

$$\nu(\circ\,\circ) = p^2 \nu(\circ * \circ) + 2p(1-p)\,\nu(\circ) + (1-p)^2 \tag{2}$$

First we consider the first upper bound $h^{(1)}(\{0\})$ on survival probability $\theta(p)$. Assuming the following Harris-FKG type correlation inequality

$$\nu(\circ\,\circ) \geqslant \nu(\circ)^2 \tag{3}$$

and using (1), we have

$$\nu(\circ) \geqslant p^2 \nu(\circ)^2 + 2p(1-p)\,\nu(\circ) + (1-p)^2 \tag{4}$$

From $\theta(p) = 1 - \nu(\circ)$, (4) can be rewritten as

$$\theta(p)\left[\theta(p) - \left\{1 - \left(\frac{1-p}{p}\right)^2\right\}\right] \leqslant 0 \tag{5}$$

Then (5) implies that for $p > p_c^*$,

$$\theta(p) \leqslant 1 - \left(\frac{1-p}{p}\right)^2$$

since if $p > p_c^*$, then $\theta(p) > 0$. This bound is equivalent to the first upper bound $h^{(1)}(\{0\})$ in Theorem 1.

Next we consider the second upper bound $h^{(2)}(\{0\})$ on survival probability $\theta(p)$ in a similar way. Assuming the following Harris-FKG type correlation inequality

$$\nu(\circ * \circ) \geqslant \nu(\circ)^2 \tag{6}$$

and using (1) and (2), we have

$$\nu(\circ) \geqslant p^2[\nu(\circ)^2 + 2p(1-p)\,\nu(\circ) + (1-p)^2] + 2p(1-p)\,\nu(\circ) + (1-p)^2 \tag{7}$$

From $\theta(p) = 1 - \nu(\{0\})$, (7) can be rewritten as,

$$\theta(p)\left[\theta(p) - \frac{2p^3 - 2p^2 + 2p - 1}{p^4}\right] \leqslant 0 \tag{8}$$

Then (8) implies that for $p > p_c^*$,

$$\theta(p) \leqslant \frac{2p^3 - 2p^2 + 2p - 1}{p^4}$$

This bound is equivalent to the second upper bound $h^{(2)}(\{0\})$ in Theorem 2.

Concerning upper bounds on survival probability $\theta(p)$, the above-mentioned derivations are not so complicated compared with those by the Harris lemma, once we assume that the Harris-FKG type correlation inequalities (3) and (6) hold for our non-attractive model (i.e., $f(2) = q = 0$ case). However at that time we could not prove these assumptions. In addition, it is known that the Harris-FKG type correlation inequalities (3) and (6) hold for the attractive case, i.e., $0 \leqslant p \leqslant q \leqslant 1$, see Belitsky *et al.*,[8] for example. So these inequalities (3) and (6) for the non-attractive case have an interest of their own. Then, under this situation, we present the following main result (i.e., correlation inequalities (3) and (6) are valid for our case). The proof of this result combines direct computations with a lower bound on a critical survival probability, $0.771 \leqslant p_c$, stated in Theorem 3.

Theorem 4. In our non-attractive model ($0 \leqslant p \leqslant 1$, $q = 0$), we have

$$\text{(a)} \quad \nu(\circ\,\circ) \geqslant \nu(\circ)^2$$

$$\text{(b)} \quad \nu(\circ * \circ) \geqslant \nu(\circ)^2$$

If $p = 0$ (resp. $p = 1$), then $\nu(\circ) = \nu(\circ\,\circ) = \nu(\circ * \circ) = 1$ (resp. $= 0$), so (a) and (b) in Theorem 4 are correct. Therefore from now on we assume that

$0<p<1$. In order to prove Theorem 4, we need the next lemma which comes from (1) and (2) by a direct computation.

Lemma 5. In our non-attractive model $(0<p<1, q=0)$, we have

$$\text{(a)} \quad \nu(\circ) - \nu(\circ\,\circ) = \left(\frac{1-p}{p}\right)^2 (1-\nu(\circ))$$

$$\text{(b)} \quad \nu(\circ) - \nu(\circ * \circ) = \left(\frac{1-p}{p}\right)^2 \left(\frac{1+p^2}{p^2}\right) (1-\nu(\circ))$$

This lemma immediately gives

Corollary 6. In our non-attractive model $(0<p<1, q=0)$, we have

$$\nu(\circ) \geqslant \nu(\circ\,\circ) \geqslant \nu(\circ * \circ)$$

Once Lemma 5 is established, part (a) comes from part (b). However we will consider the general $f(2)=q\in[0,1]$ case in the final part of this letter and refer an argument in the proof of part (a). So we do not omit the proof of part (a).

Proof of Part (a): $\nu(\circ\,\circ) \geqslant \nu(\circ)^2$. From Lemma 5 (a), we have

$$\nu(\circ) = \frac{\nu(\circ\,\circ)+\alpha}{1+\alpha} \tag{9}$$

where $\alpha=(1-p)^2/p^2$. So by using (9), the inequality we want to show can be rewritten as

$$(1-\nu(\circ\,\circ))(\nu(\circ\,\circ)-\alpha^2) \geqslant 0$$

Since $\nu(\circ\,\circ)\in[0,1]$, it suffices to show

$$\nu(\circ\,\circ) \geqslant \alpha^2$$

We easily see that (2) implies

$$\nu(\circ\,\circ) \geqslant (1-p)^2$$

So it is enough to show that

$$(1-p)^2 \geqslant \alpha^2 = \left(\frac{1-p}{p}\right)^4$$

This inequality is equivalent to

$$\{p^2+(1-p)\}\{p^2-(1-p)\} \geqslant 0$$

So if $p^2+p-1 \geqslant 0$, then $\nu(\circ\,\circ) > \nu(\circ)^2$. Therefore if $p \geqslant (-1+\sqrt{5})/2 = 0.618...$, then $\nu(\circ\,\circ) \geqslant \nu(\circ)^2$. On the other hand, Theorem 3 implies that if $p \leqslant 0.771$, then $\nu(\circ) = 1$. By Lemma 5 (a), we have $\nu(\circ\,\circ) = 1$. Then $\nu(\circ\,\circ) \geqslant \nu(\circ)^2$ is trivial. Therefore we prove

$$\nu(\circ\,\circ) \geqslant \nu(\circ)^2$$

for any $p \in [0, 1]$.

Proof of Part (b): $\nu(\circ * \circ) \geqslant \nu(\circ)^2$. By the definition of this model, we have

$$\begin{aligned}\nu(\circ * \circ) = p^4\nu(\circ\circ\circ\circ) &+ 2p^3(1-p)\,\nu(\circ\circ\circ)\\ &+ 2p^3(1-p)\,\nu(\circ\circ*\circ) + 3p^2(1-p)^2\,\nu(\circ\,\circ)\\ &+ 2p^2(1-p)^2\,\nu(\circ*\circ) + p^2(1-p)^2\,\nu(\circ**\circ)\\ &+ 4p(1-p)^3\,\nu(\circ) + (1-p)^4\end{aligned}$$

Each term of the right-hand side can be bounded below by the following:

$$\begin{aligned}\nu(\circ * \circ) &= p^4\nu(\circ\circ\circ\circ)\\ &\quad (\geqslant p^4\{(1-p)^2+2p(1-p)^3+p^2(1-p)^4\})\\ &\qquad + 2p^3(1-p)\,\nu(\circ\circ\circ)\\ &\quad (\geqslant 2p^3(1-p)\{(1-p)^2+2p(1-p)^3+p^2(1-p)^4\})\\ &\qquad + 2p^3(1-p)\,\nu(\circ\circ*\circ)\\ &\quad (\geqslant 2p^3(1-p)(1-p)^4)\\ &\qquad + 3p^2(1-p)^2\,\nu(\circ\,\circ)\\ &\quad (\geqslant 3p^2(1-p)^2\,\{(1-p)^2+2p(1-p)^3+p^2(1-p)^4\})\\ &\qquad + 2p^2(1-p)^2\,\nu(\circ*\circ)\\ &\quad (\geqslant 2p^2(1-p)^2\,\{(1-p)^4+p^4(1-p)^2\})\\ &\qquad + p^2(1-p)^2\,\nu(\circ**\circ)\\ &\quad (\geqslant p^2(1-p)^2\,(1-p)^4)\\ &\qquad + 4p(1-p)^3\,\nu(\circ)\\ &\quad (\geqslant 4p(1-p)^3\,\{p^2(1-p)^2+2p(1-p)^3+(1-p)^2\}+(1-p)^4)\\ &\qquad + (1-p)^4\\ &\quad (=(1-p)^4)\end{aligned}$$

As in a similar fashion of the proof of part (a), we confirm that $\nu(\circ * \circ) \geqslant \beta^2$ gives $\nu(\circ * \circ) \geqslant \nu(\circ)^2$, where $\beta = \alpha(1+p^2)/p^2$. Then we can check that the following inequality holds for any $p \geqslant 0.77077...$,

$$\begin{aligned} & p^4\{(1-p)^2 + 2p(1-p)^3 + p^2(1-p)^4\} \\ &\quad + 2p^3(1-p)\{(1-p)^2 + 2p(1-p)^3 + p^2(1-p)^4\} \\ &\quad + 2p^3(1-p)(1-p)^4 \\ &\quad + 3p^2(1-p)^2\{(1-p)^2 + 2p(1-p)^3 + p^2(1-p)^4\} \\ &\quad + 2p^2(1-p)^2\{(1-p)^4 + p^4(1-p)^2\} \\ &\quad + p^2(1-p)^2(1-p)^4 \\ &\quad + 4p(1-p)^3\{p^2(1-p)^2 + 2p(1-p)^3 + (1-p)^2\} \\ &\quad + (1-p)^4 \\ & \geqslant \beta^2 = \frac{(1-p)^4(1+p^2)^2}{p^8} \end{aligned}$$

Note that the left-hand side is bounded above by $\nu(\circ * \circ)$. On the other hand, Theorem 3 implies that if $p \leqslant 0.771$, then $\nu(\circ) = 1$. From Lemma 5 (b), we have $\nu(\circ * \circ) = 1$. Then $\nu(\circ * \circ) \geqslant \nu(\circ)^2$ is correct. So we obtain

$$\nu(\circ * \circ) \geqslant \nu(\circ)^2$$

for any $p \in [0, 1]$.

Finally we consider the case of general $q \in [0, 1]$. If $p \leqslant 0.5$ and $q \in [0, 1]$, then $\nu(\circ) = \nu(\circ\,\circ) = 1$ by comparison with a branching process and Lemma 5 (a). So $\nu(\circ\,\circ) \geqslant \nu(\circ)^2$ holds in this region. Therefore we assume that $p > 0.5$ and $q \in [0, 1]$. From the definition of the model, we have

$$\begin{aligned} \nu(\circ\,\circ) &= p^2 q \nu(\circ\,\circ\,\circ) + 2pq(1-p)\,\nu(\circ\,\circ) + p^2(1-q)\,\nu(\circ * \circ) \\ &\quad + \{2p(1-p)(1-q) + q(1-p)^2\}\,\nu(\circ) + (1-p)^2(1-q) \end{aligned}$$

From Corollary 6 (i.e., $\nu(\circ) \geqslant \nu(\circ\,\circ)$) and $\nu(\circ\,\circ\,\circ) \geqslant 0$, we obtain

$$\{1 - 2pq(1-p) - 2p(1-p)(1-q) - q(1-p)^2\}\,\nu(\circ\,\circ) \geqslant (1-p)^2(1-q)$$

That is,

$$\{p^2 + (1-q)(1-p)^2\}\,\nu(\circ\,\circ) \geqslant (1-p)^2(1-q)$$

Then we have

$$\nu(\circ\,\circ) \geqslant \frac{(1-p)^2\,(1-q)}{p^2+(1-q)(1-p)^2}$$

In the previous argument, $\nu(\circ\,\circ) \geqslant \alpha^2$ gives $\nu(\circ\,\circ) \geqslant \nu(\circ)^2$, so it suffices to show that

$$\frac{(1-p)^2\,(1-q)}{p^2+(1-q)(1-p)^2} \geqslant \frac{(1-p)^4}{p^4}$$

Therefore $\nu(\circ\,\circ) \geqslant \nu(\circ)^2$ holds for the following region (see Fig. 1):

$$q \leqslant 1 - \frac{p^2(1-p)^2}{p^4-(1-p)^4}$$

This result suggests that

$$\nu(\circ\,\circ) \geqslant \nu(\circ)^2$$

holds even in all non-attractive region ($q<p$). Furthermore we extend this conjecture as follows:

Conjecture 7. Even in the non-attractive case ($0 \leqslant q < p \leqslant 1$), we have

$$\nu(A \cup B) \geqslant \nu(A)\,\nu(B) \qquad \text{for any finite} \quad A, B \subset 2\mathbf{Z}$$

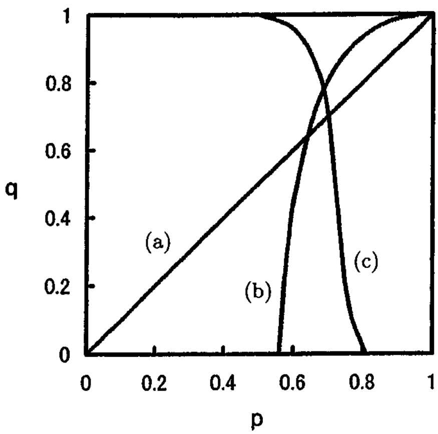

Fig. 1. The line (a) is the graph of $q=p$. The line (b) is the graph of $q=1-(p^2(1-p)^2)/(p^4-(1-p)^4)$. The line (c) is an estimated critical line, for example, given by Kinzel.[3]

Note that Conjecture 7 is valid for the attractive case ($0 \leqslant p \leqslant q \leqslant 1$), see Belitsky *et al.*,[9] for example. We think that to prove this conjecture is one of the future interesting problems.

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